

# A NEW VERSION OF HOMOTOPICAL HAUSDORFF

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ABSTRACT. It is known that shape injectivity implies homotopical Hausdorff and that the converse does not hold, even if the space is required to be a Peano continuum. This paper gives an alternative definition of homotopical Hausdorff inspired by a new topology on the set of fixed endpoint homotopy classes of paths. This version is equivalent to shape injectivity for Peano spaces.

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## 1. INTRODUCTION

Homotopical Hausdorff is a homotopical criterion that detects if a Hausdorff space  $X$  has  $\tilde{X}$  Hausdorff where  $\tilde{X}$  is the set of fixed endpoint homotopy classes of paths in  $X$  starting at some basepoint with a standard topology (see Definition 3.1). Homotopical Hausdorff was discussed in [14] and [1] and given its present name in [4]. In [8] the authors show that shape injectivity implies homotopical Hausdorff. A space  $X$  is shape injective if the homomorphism  $\pi_1(X) \rightarrow \tilde{\pi}_1(X)$  is injective. Spaces that are known to be shape injective include one dimensional Hausdorff compacta [7] and subsets of closed surfaces [9]. Thus these spaces have nice spaces of path homotopy classes.

The authors in [5] give examples of two Peano continua that are homotopically Hausdorff but not shape injective. The present paper notes that if the definition of homotopical Hausdorff is modified to reflect a new topology on  $\tilde{X}$  (Definition 3.3) then the two concepts are equivalent for Peano spaces (Theorem 3.7).

Section 2 gives a treatment of shape injectivity that mirrors the theory of generalized paths in [2]. This viewpoint relates paths in a space  $X$  to chains of points in  $X$ . It is quite geometric and lends itself to the proof of Theorem 3.7.

## 2. SHAPE INJECTIVITY

Generalized paths and the uniform shape group are defined for uniform spaces in [2]. We introduce an analogous construction for all topological spaces. Let us first recall the definition of the classical shape group.

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We will consider only normal open covers. Recall an open cover  $\mathcal{U}$  of  $X$  is normal if it admits a partition of unity  $\{\phi_U : X \rightarrow [0, 1]\}_{U \in \mathcal{U}}$  with  $\phi_U^{-1}(0, 1] \subset U$  for each  $U \in \mathcal{U}$ . We say that the partition of unity is subordinate to  $\mathcal{U}$ . The partition of unity can be chosen to be locally finite. The nerve  $N(\mathcal{U})$  of  $\mathcal{U}$  is the simplicial complex whose vertices are elements of  $\mathcal{U}$  and whose simplices are finite subsets  $\mathcal{A} \subset \mathcal{U}$  such that the intersection of the elements of  $\mathcal{A}$  is nonempty.

Fix an  $x_0 \in X$  and for each normal cover  $\mathcal{U}$  fix a  $U_0 \in \mathcal{U}$  with  $x_0 \in U_0$ . Given two normal covers  $\mathcal{U}$  and  $\mathcal{V}$ , define  $\mathcal{U} \leq \mathcal{V}$  if  $(\mathcal{V}, V_0)$  refines  $(\mathcal{U}, U_0)$ , that is,  $\mathcal{V}$  refines  $\mathcal{U}$  and  $V_0 \subset U_0$ . In this case choose a bonding map  $p : N(\mathcal{V}) \rightarrow N(\mathcal{U})$  such that each  $V \in \mathcal{V}$  gets mapped to a  $U \in \mathcal{U}$  with  $V \subset U$ , making sure to send  $V_0$  to  $U_0$ . The shape group  $\tilde{\pi}_1(X, x_0)$  is the inverse limit  $\varprojlim \pi_1(N(\mathcal{U}), U_0)$ . [10]

Given a cover  $\mathcal{U}$  of  $X$  define a  $\mathcal{U}$ -chain to be a finite list  $x_1, \dots, x_n$  of points in  $X$  such that for each  $i < n$ ,  $x_i, x_{i+1} \in U$  for some  $U \in \mathcal{U}$ . Let  $R(X, \mathcal{U})$  be the simplicial complex whose vertices are points in  $X$  and  $A \subset X$  is a simplex if it is a finite subset of some  $U \in \mathcal{U}$ . It is the Rips complex of  $X$  with respect to  $\mathcal{U}$ .

We identify a  $\mathcal{U}$ -chain  $x_1, \dots, x_n$  with the concatenation of the edge paths  $[x_1, x_2], \dots, [x_{n-1}, x_n]$  in  $R(X, \mathcal{U})$ . We define two  $\mathcal{U}$ -chains to be  $\mathcal{U}$ -homotopic if the corresponding paths in  $R(X, \mathcal{U})$  are fixed endpoint homotopic. This homotopy can be chosen to be simplicial. Thus two  $\mathcal{U}$ -chains are  $\mathcal{U}$ -homotopic if one can move from one to the other by a finite sequence of vertex additions and deletions.

Define a generalized path in  $X$  to be a collection  $\alpha = \{[\alpha_U]_{\mathcal{U}}\}$  of equivalence classes of  $\mathcal{U}$ -chains in  $X$ , where  $\mathcal{U}$  runs over all normal open covers of  $X$ , such that if  $\mathcal{V}$  refines  $\mathcal{U}$ ,  $\alpha_{\mathcal{V}}$  is  $\mathcal{U}$ -homotopic to  $\alpha_{\mathcal{U}}$ . We define the covering shape group  $\tilde{\pi}_1^{\text{cov}}(X, x_0)$  to be the group of generalized loops in  $X$  based at  $x_0$  under the operation of concatenation. It is isomorphic to  $\varprojlim \pi_1(R(X, \mathcal{U}), x_0)$ .

We will show that  $\tilde{\pi}_1^{\text{cov}}(X, x_0)$  is isomorphic to the classical shape group. In order to do so, let us recall the following definition. Given an open cover  $\mathcal{U}$  of  $X$ , the star of a point  $x \in X$  in  $\mathcal{U}$  is the union of all  $U \in \mathcal{U}$  containing  $x$ . We say that a cover  $\mathcal{V}$  is a star refinement of a cover  $\mathcal{U}$  if the cover  $\{\text{St}(x, \mathcal{V}) : x \in X\}$  refines  $\mathcal{U}$ . Any normal open cover has a normal star refinement. [6, Proposition 5.3].

It is more convenient to use the following notion of a star of a cover. Given an open cover  $\mathcal{U}$  and a  $U \in \mathcal{U}$ , let  $\text{St}U$  be the union of all  $V \in \mathcal{U}$  that meet  $U$ . Let  $\text{St}\mathcal{U}$  be the set of all  $\text{St}U$  for  $U \in \mathcal{U}$ . Notice the similarity between the open set  $\text{St}U$  in  $X$  and the open star  $\text{St}U$  of the vertex  $U$  in  $N(\mathcal{U})$ . They are both defined in terms of all  $V \in \mathcal{U}$  that meet  $U$ .

**Lemma 2.1.** *Suppose  $\mathcal{W}$  is a star refinement of  $\mathcal{V}$  and that  $\mathcal{V}$  is a star refinement of  $\mathcal{U}$ . Then  $\text{St}\mathcal{W}$  refines  $\mathcal{U}$ .*

*Proof.* Given  $W \in \mathcal{W}$ , let  $x \in W$ . We will show that  $\text{St}(W, \mathcal{W}) \subset \text{St}(x, \mathcal{V})$ . Suppose  $y \in \text{St}(W, \mathcal{W})$ . Then  $y \in W'$  where  $W' \in \mathcal{W}$  meets  $W$ , say at a point  $z$ . Then  $x, y \in \text{St}(z, \mathcal{W})$  which is contained in some  $V \in \mathcal{V}$ . Thus  $y \in \text{St}(x, \mathcal{V})$ .  $\square$

**Proposition 2.2.**  *$\tilde{\pi}_1(X, x_0)$  is isomorphic to  $\tilde{\pi}_1^{\text{cov}}(X, x_0)$ .*

*Proof.* Fix a basepoint  $x_0 \in X$  and for each normal cover  $\mathcal{U}$  of  $X$ , fix a “basepoint”  $U_0 \in \mathcal{U}$  with  $x_0 \in U_0$ . Define a pointed map  $(R(X, \mathcal{U}), x_0) \rightarrow (N(\text{St}\mathcal{U}), \text{St}U_0)$  to send a vertex  $x \in X$  to  $\text{St}U$  for some  $U \in \mathcal{U}$  with  $x \in U$ . Note we can assume  $\text{St}U_0$  is the basepoint of  $\text{St}\mathcal{U}$  since any other  $\text{St}U$  that contains  $x_0$  meets  $\text{St}U_0$ . Let us see that this map on vertices extends to a simplicial map. Suppose  $[x_1, \dots, x_n] \in R(X, \mathcal{U})$ . If  $x_i \mapsto \text{St}U_i$ , then  $x_1 \in \text{St}U_i$  for each  $i \leq n$  so  $[\text{St}U_1, \dots, \text{St}U_n] \in N(\text{St}\mathcal{U})$ .

Now define a pointed map  $(N(\mathcal{U}), U_0) \rightarrow (R(X, \text{St}\mathcal{U}), x_0)$  to send a vertex  $U \in \mathcal{U}$  to some  $x$  with  $x \in U$ . Let us see that this map extends to a simplicial map. Suppose  $[U_1, \dots, U_n] \in N(\mathcal{U})$ . If  $U_i \mapsto x_i$ , then  $x_1, \dots, x_n \in \text{St}U_1$  so  $[x_1, \dots, x_n] \in R(X, \text{St}\mathcal{U})$ .

These maps induce homomorphisms  $\pi_1(R(X, \mathcal{U}), x_0) \rightarrow \pi_1(N(\text{St}\mathcal{U}), \text{St}U_0)$  and  $\pi_1(N(\mathcal{U}), U_0) \rightarrow \pi_1(R(X, \text{St}\mathcal{U}), x_0)$ . By the lemma and the fact that any normal open cover has a normal star refinement, it suffices to check that the following two diagrams commute.

$$\begin{array}{ccc}
 & \pi_1(R(X, \mathcal{U}), x_0) & \\
 & \swarrow & \downarrow \\
 \pi_1(N(\text{St}\mathcal{U}), \text{St}U_0) & & \pi_1(R(X, \text{St}\text{St}\mathcal{U}), x_0) \\
 & \searrow & \\
 & \pi_1(N(\mathcal{U}), U_0) & \\
 & \downarrow & \swarrow \\
 & \pi_1(N(\text{St}\text{St}\mathcal{U}), \text{St}\text{St}U_0) & \pi_1(R(X, \text{St}\mathcal{U}), x_0)
 \end{array}$$

Suppose  $x_0, \dots, x_n$  is a  $\mathcal{U}$ -chain in  $X$  representing a loop in  $R(X, \mathcal{U})$  based at  $x_0$ . Suppose  $x_0, \dots, x_n$  is sent to  $\text{St}U_0, \dots, \text{St}U_n$  which in turn is sent to  $y_0, \dots, y_n$ . Now  $y_0 = y_n = x_0$  by assumption. To see that  $y_0, \dots, y_n$  is  $\text{St}\text{St}\mathcal{U}$ -homotopic to  $x_0, \dots, x_n$ , notice that for any  $i < n$ ,  $x_i, y_i, x_{i+1}, y_{i+1} \in \text{St}\text{St}U$  where  $U \in \mathcal{U}$  contains  $x_i$  and  $x_{i+1}$ .

Now suppose  $U_0, \dots, U_n$  is a sequence of vertices in  $N(\mathcal{U})$  that represents a loop in  $N(\mathcal{U})$  based at  $U_0$ . Suppose  $U_0, \dots, U_n$  is sent to  $x_0, \dots, x_n$  which in turn is sent to  $\text{St}\text{St}V_0, \dots, \text{St}\text{St}V_n$  where  $V_i \in \mathcal{U}$ . Now  $V_0 = V_n = U_0$  by assumption. To see that the loop represented by  $\text{St}\text{St}V_0, \dots, \text{St}\text{St}V_n$  is homotopic to the one represented by  $U_0, \dots, U_n$  in  $N(\text{St}\text{St}\mathcal{U})$ , notice that for any  $i < n$ ,  $x \in \text{St}\text{St}U_i \cap \text{St}\text{St}V_i \cap \text{St}\text{St}U_{i+1} \cap \text{St}\text{St}V_{i+1}$  where  $x$  is an element in  $U_i \cap U_{i+1}$ .  $\square$

There is a natural homomorphism  $\pi_1(X, x_0) \rightarrow \tilde{\pi}_1(X, x_0)$  from the fundamental group to the shape group. Suppose  $\alpha$  is a path in  $X$  and  $\mathcal{U}$  is an open cover of  $X$ . Choose  $\delta > 0$  so that any subinterval of  $[0, 1]$  of length  $\delta$  is sent by  $\alpha$  to some  $U \in \mathcal{U}$ . Define a  $\mathcal{U}$ -chain  $\varphi_{\mathcal{U}}(\alpha) = \alpha(0), \alpha(\delta), \alpha(2\delta), \dots, \alpha(1)$ . This definition is independent of the choice of  $\delta$ ; given  $\delta_1 < \delta_2$ , the corresponding  $\mathcal{U}$ -chains will be  $\mathcal{U}$ -homotopic. Simply add the two chains together according to the order on  $[0, 1]$  to get another  $\mathcal{U}$ -chain which is  $\mathcal{U}$ -homotopic to both.

A similar argument shows that if  $\mathcal{V}$  refines  $\mathcal{U}$ ,  $\varphi_{\mathcal{V}}(\alpha)$  will be  $\mathcal{U}$ -homotopic to  $\varphi_{\mathcal{U}}(\alpha)$ . Thus we have a generalized path  $\varphi(\alpha) = \{\varphi_{\mathcal{U}}(\alpha)\}$ . We show that  $\varphi$  induces a well-defined homomorphism  $\pi_1(X, x_0) \rightarrow \tilde{\pi}_1(X, x_0)$ . Suppose  $\alpha$  is fixed endpoint homotopic to  $\beta$ . Let  $\mathcal{U}$  be a cover of  $X$  and  $\delta > 0$  be such that any square

$I \times I \subset [0, 1] \times [0, 1]$  of side length  $\delta$  is sent by the homotopy to an element of  $\mathcal{U}$ . We have a sequence of paths  $\alpha_0, \alpha_\delta, \alpha_{2\delta}, \dots, \beta$  given by the homotopy. To see that  $\varphi_{\mathcal{U}}(\alpha_{i\delta})$  is  $\mathcal{U}$ -homotopic to  $\varphi_{\mathcal{U}}(\alpha_{(i+1)\delta})$ , notice the following chain is  $\mathcal{U}$ -homotopic to both.

$$\alpha_{i\delta}(0) \ \alpha_{i\delta}(\delta) \ \alpha_{(i+1)\delta}(\delta) \ \alpha_{(i+1)\delta}(2\delta) \ \alpha_{i\delta}(2\delta) \ \alpha_{i\delta}(3\delta) \ \alpha_{(i+1)\delta}(3\delta) \cdots$$

We end the section by showing that this homomorphism is identical to the classical homomorphism. Given a cover  $\mathcal{U}$ , there is a map  $\phi : X \rightarrow N(\mathcal{U})$  given by a partition of unity subordinate to  $\mathcal{U}$ ,  $\phi(x) = \sum \phi_U(x)U$ . This map enjoys the property that  $\phi^{-1}(\text{St}U) \subset U$  where  $\text{St}U$  is the open star of the vertex  $U$  in  $N(\mathcal{U})$ .  $\phi$  induces a homomorphism  $\pi_1(X) \rightarrow \pi_1(N(\mathcal{U}))$ . We show the following diagram commutes.

$$\begin{array}{ccc} & \pi_1(R(X, \mathcal{U}), U_0) & \\ \nearrow \pi_1(X, x_0) & \downarrow & \\ & \pi_1(N(\text{St}\mathcal{U}), \text{St}U_0) & \end{array}$$

Let  $\phi : X \rightarrow N(\text{St}\mathcal{U})$  be a map given by a partition of unity subordinate to  $\text{St}\mathcal{U}$ . Let  $\alpha$  be a path in  $X$ . Let  $\delta > 0$  so that if  $I$  is a subinterval of  $[0, 1]$  of length  $\delta$ ,  $\alpha(I)$  is sent by  $\phi$  to some  $\text{St}U$ , the open star of a vertex  $U$  of  $N(\text{St}\mathcal{U})$ . Let  $\delta$  also be such that  $\alpha(I)$  lies in an element of  $\mathcal{U}$ .

Now  $\alpha$  is sent to the chain  $\alpha(0), \alpha(\delta), \alpha(2\delta), \dots$  in  $R(X, \mathcal{U})$  which in turn is sent to the chain  $\text{St}U_0, \text{St}U_\delta, \text{St}U_{2\delta}, \dots$  in  $N(\text{St}\mathcal{U})$  where  $\alpha(k\delta) \in U_{k\delta}$  for  $k \geq 0$ . On the other hand,  $\alpha$  is sent to the path  $\phi\alpha$  in  $N(\text{St}\mathcal{U})$ . We need to show that  $\phi\alpha$  is fixed endpoint homotopic to the concatenation of edge paths associated with the chain  $\text{St}U_0, \text{St}U_\delta, \text{St}U_{2\delta}, \dots$ . We proceed by induction on the number of terms in this chain.

Now  $\phi\alpha[0, \delta] \subset \text{St}U$  and  $\phi\alpha[\delta, 2\delta] \subset \text{St}U_1$  for some  $U, U_1 \in \mathcal{U}$ . Since  $\phi\alpha(\delta) \in \text{St}U \cap \text{St}U_1$ , it is in an open simplex having  $U$  and  $U_1$  as vertices. Now  $\alpha[0, \delta] \subset \text{St}U$  and  $\alpha[\delta, 2\delta] \subset \text{St}U_1$  so  $\alpha(\delta) \in \text{St}U \cap \text{St}U_1$  and  $[\text{St}U, \text{St}U_1, \text{St}U_\delta]$  is a simplex. The open simplex  $[\text{St}U, \text{St}U_1, \text{St}U_\delta]$  is contained in  $\text{St}U \cap \text{St}U_1$  so we can join  $\phi\alpha(\delta)$  to  $\text{St}U_\delta$  by a path  $\gamma$  with  $\gamma[0, 1] \subset \text{St}U \cap \text{St}U_1$ . Then, since  $\phi\alpha[0, \delta] \subset \text{St}U$ , we can find a homotopy from  $\phi\alpha[0, \delta]$  to the edge path  $[\text{St}U_0, \text{St}U_\delta]$  that fixes  $\alpha(0) = \text{St}U_0$  and follows  $\gamma$  from  $\phi\alpha(\delta)$  to  $\text{St}U_\delta$ .

For each  $i > 1$ ,  $\phi\alpha[i\delta, (i+1)\delta] \subset \text{St}U_i$  for some  $U_i \in \mathcal{U}$ . Suppose that  $\phi\alpha[0, k\delta]$  is homotopic to the concatenation of edge paths associated with  $\text{St}U_0, \text{St}U_\delta, \dots, \text{St}U_{k\delta}$  where the homotopy fixes  $\alpha(0) = \text{St}U_0$  and follows a path  $\gamma_k$  from  $\phi\alpha(k\delta)$  to  $\text{St}U_{k\delta}$  with  $\gamma_k[0, 1] \subset \text{St}U_{k-1} \cap \text{St}U_k$ . We follow the same procedure as above to find a path  $\gamma_{k+1}$  from  $\phi\alpha((k+1)\delta)$  to  $\text{St}U_{(k+1)\delta}$  that is contained in  $\text{St}U_k \cap \text{St}U_{k+1}$ . Since  $\phi\alpha[k\delta, (k+1)\delta] \subset \text{St}U_k$ , we can find a homotopy from  $\phi\alpha[k\delta, (k+1)\delta]$  to the edge path  $[\text{St}U_{k\delta}, \text{St}U_{(k+1)\delta}]$  that follows  $\gamma_k$  from  $\phi\alpha(k\delta)$  to  $\text{St}U_{k\delta}$  and  $\gamma_{k+1}$  from  $\phi\alpha((k+1)\delta)$  to  $\text{St}U_{(k+1)\delta}$ .

## 3. HOMOTOPICAL HAUSDORFF

We recall a standard topology on  $\tilde{X}$ , the set of fixed endpoint homotopy classes of paths in  $X$  starting at some basepoint  $x_0$ .

**Definition 3.1.** Given  $[\alpha] \in \tilde{X}$  with terminal point  $x$  and a neighborhood  $U$  of  $x$  in  $X$ ,  $B([\alpha], U)$  is the set of all  $[\beta] \in \tilde{X}$  such that  $\alpha^{-1}\beta$  is fixed endpoint homotopic to a path in  $U$ . We will call the topology generated by these sets the whisker topology on  $\tilde{X}$  following [3].

This topology is used in Spanier [13] and Munkres [11] for the classic construction of covering spaces. It is equivalent to the quotient topology inherited from  $(X, x_0)^{(I, 0)}$  under the compact open topology for locally path connected and semilocally simply connected spaces [8, Lemma 2.1].

Investigations into the structure of  $\tilde{X}$  leads one to realize that it can fail to be Hausdorff. The harmonic archipelago in [1] is a standard example of a Hausdorff space whose space of path homotopy classes is not Hausdorff. This situation motivates the following definition (see [4]).

**Definition 3.2.** A space  $X$  is homotopically Hausdorff if for each  $x \in X$  and each essential loop  $\gamma$  based at  $x$ , there is a neighborhood  $U$  of  $x$  such that  $\gamma$  is not fixed endpoint homotopic to a path in  $U$ .

Notice that a space  $X$  is homotopically Hausdorff if and only if for all  $x \in X$ ,  $\cap \pi_1(U, x) = 1$  where  $U$  runs over all neighborhoods of  $x$ . Also, for a Hausdorff space  $X$ ,  $X$  is homotopically Hausdorff if and only if  $\tilde{X}$  is Hausdorff for any basepoint.

It is shown in [8] that if a space is shape injective then it is homotopically Hausdorff. The space  $A$  in [5] is an example of a Peano continuum that is homotopically Hausdorff but not shape injective (see Example 3.5).

Investigation into the structure of  $\pi_1(X)$  as a subspace of  $\tilde{X}$  in [3] lead to the definition of a new topology on  $\tilde{X}$ . The new topology is based on the following definition. Given an open cover  $\mathcal{U}$ , let  $\pi(\mathcal{U}, x)$  be the subgroup of  $\pi_1(X, x)$  generated by elements of the form  $[\alpha\gamma\alpha^{-1}]$  where  $\alpha$  is a path starting at  $x$  and  $\gamma$  is a loop in some  $U \in \mathcal{U}$ . These groups are used in Spanier [13] to detect when a fibration with unique path lifting is a covering map.

**Definition 3.3.** Given  $[\alpha] \in \tilde{X}$  with terminal point  $x$ , a normal open cover  $\mathcal{U}$  of  $X$ , and a neighborhood  $V$  of  $x$  in  $X$ ,  $B([\alpha], \mathcal{U}, V)$  is the set of all  $[\beta] \in \tilde{X}$  such that  $\alpha^{-1}\beta$  is fixed endpoint homotopic to a loop in  $\pi(\mathcal{U}, x)$  concatenated with a path in  $V$ . We will call the topology generated by these basic sets the lasso topology.

There are slight differences between the above definition and the one that appears in [3]. As in the definition of the shape group, here we restrict our attention to normal covers. Also, in [3] it is required that  $V \in \mathcal{U}$ . This requirement does not effect the topology generated by the sets.

We now define an analogous version of homotopical Hausdorff for the lasso topology.

**Definition 3.4.** A space  $X$  is lasso homotopically Hausdorff if for each  $x \in X$  and each essential loop  $\gamma$  based at  $x$ , there is a normal open cover  $\mathcal{U}$  of  $X$  such that  $[\gamma] \notin \pi_1(\mathcal{U}, x)$ .

Notice that a space  $X$  is lasso homotopically Hausdorff if for all  $x \in X$ ,  $\cap \pi_1(\mathcal{U}, x) = 1$  where  $\mathcal{U}$  runs over all normal open covers of  $X$ . Also, for a Hausdorff space  $X$ ,  $X$  is lasso homotopically Hausdorff if and only if  $\tilde{X}$  is Hausdorff for any basepoint under the lasso topology.

This concept was investigated in [8] where it is shown that if  $\cap \pi_1(\mathcal{U}) = 1$  for some collection of open covers of  $X$ , then the endpoint map  $\tilde{X} \rightarrow X$  has unique path lifting (where  $\tilde{X}$  is given the whisker topology).

**Example 3.5.** We show that the space  $A$  from [5] is not lasso homotopically Hausdorff. Let  $A'$  be the topologist's sine curve  $\{(x, \sin(1/x)) : x \in (0, 1]\} \cup \{0\} \times [-1, 1]$  rotated about its limiting arc. It is the *surface* and *central limit arc* portions of  $A$ . *Connecting arcs* are added to form the Peano continuum  $A$  and the authors show that if a loop in  $A'$  is nullhomotopic in  $A$  then it is nullhomotopic in  $A'$  (Lemma 4.3). Choose a basepoint  $x$  on the surface portion of  $A$  and a loop  $\beta$  that goes once around the surface portion. Since  $\beta$  is essential in  $A'$  it must be essential in  $A$  as well. Given any neighborhood  $U$  of a point on the central limit arc,  $\beta$  is fixed endpoint homotopic to a loop of the form  $\alpha\gamma\alpha^{-1}$  where  $\gamma$  is contained in  $U$ . Thus  $[\beta] \in \pi_1(\mathcal{U}, x)$  for all open covers  $\mathcal{U}$  of  $A$ . Note that  $A$  is not shape injective.

We now see that this version of homotopical Hausdorff is equivalent to shape injectivity for Peano spaces.

**Lemma 3.6.** *Suppose  $X$  is locally path connected. Any normal open cover  $\mathcal{U}$  of  $X$  has a normal open refinement composed of path connected sets.*

*Proof.* Let  $\mathcal{U}$  be an open cover with associated partition of unity  $\{\phi_U\}$ . Given  $U \in \mathcal{U}$ , decompose it into its path components  $\{V_\alpha\}$ . Since  $X$  is locally path connected these components are open. Given  $x \in X$ , define  $\psi_{V_\alpha}(x) = \phi_U(x)$  if  $x \in V_\alpha$  and  $\psi_{V_\alpha}(x) = 0$  otherwise. Then  $\{\psi_{V_\alpha}\}_{U \in \mathcal{U}}$  is a partition of unity subordinate to  $\{V_\alpha\}_{U \in \mathcal{U}}$ .  $\square$

**Theorem 3.7.** *Suppose  $X$  is path connected and locally path connected. Then  $X$  is lasso homotopically Hausdorff if and only if it is shape injective.*

*Proof.* The reverse direction is essentially [8, Proposition 4.8]. We provide a proof here. Suppose  $X$  is shape injective and that  $[\beta] \in \cap \pi_1(\mathcal{U}, x)$ . Given  $\mathcal{U}$ , let  $\lambda$  be a path fixed endpoint homotopic to  $\beta$  so that  $\lambda = \lambda_1 \cdots \lambda_n$  where each  $\lambda_i = \alpha_i \gamma_i \alpha_i^{-1}$ ,  $\alpha_i$  is a path starting at  $x$ , and  $\gamma_i$  is a loop in some  $U \in \mathcal{U}$ . Send  $\lambda$  to  $\pi_1(R(X, \mathcal{U}), x)$ . Then the image of  $\gamma_i$  is  $\mathcal{U}$ -homotopic to the constant chain at the terminal point of  $\alpha_i$  so the image of  $\lambda_i$  is  $\mathcal{U}$ -homotopic to the constant chain at  $x$ . Thus the image of  $\lambda$  is trivial and the image of  $\lambda$  in  $\tilde{\pi}_1(X, x)$  is trivial. Given that  $X$  is shape injective, we have that  $\lambda$  is trivial.

Now suppose that  $X$  is lasso homotopically Hausdorff. Suppose  $\beta$  is a loop in  $X$  based at  $x$  whose image in  $\tilde{\pi}_1(X, x)$  is trivial. Given a cover  $\mathcal{U}$ , we wish to show  $[\beta] \in \pi_1(\mathcal{U}, x)$ . Let  $\mathcal{V}$  be a cover so that  $\text{St}\mathcal{V}$  refines  $\mathcal{U}$  and let  $\mathcal{W}$  be a refinement of  $\mathcal{V}$  composed of path connected sets. The image of  $\beta$  in  $R(X, \mathcal{W})$  is  $\mathcal{W}$ -homotopic to the trivial chain at  $x$ . We proceed by induction on the number of steps in this simplicial homotopy.

The image of  $\beta$  in  $R(X, \mathcal{W})$  is represented by a  $\mathcal{W}$ -chain  $x_0, \dots, x_n$ , i.e.,  $\beta = \beta_0 \cdots \beta_{n-1}$  where each  $\beta_i$  is a path in some element of  $\mathcal{W}$  from  $x_i$  to  $x_{i+1}$ .

Suppose a step of the simplicial homotopy starts at the  $\mathcal{W}$ -chain  $y_0, \dots, y_m$ . Suppose there is a  $[\lambda] \in \pi_1(\mathcal{U}, x)$  and a path  $\alpha = \alpha_0 \cdots \alpha_{m-1}$  associated with

$y_0, \dots, y_m$  (that means each  $\alpha_i$  is a path in some element of  $\mathcal{W}$  from  $y_i$  to  $y_{i+1}$ ) such that  $\beta$  is fixed endpoint homotopic to  $\lambda\alpha$ . We show the same thing can be said about the next chain in the simplicial homotopy.

Suppose the next step of the simplicial homotopy is obtained by vertex addition, say  $\dots, y_i, y_{i+1}, \dots$  to  $\dots, y_i, y, y_{i+1}, \dots$ . Now  $y_i, y, y_{i+1} \in W$  for some  $W \in \mathcal{W}$ . Join  $y_i$  to  $y$  by a path  $\lambda_1$  in  $W$  and join  $y$  to  $y_{i+1}$  by a path  $\lambda_2$  in  $W$ . Let  $\lambda = \alpha_0 \cdots \alpha_{i+1} \lambda_2^{-1} \lambda_1^{-1} \alpha_i^{-1} \cdots \alpha_0^{-1}$  and  $\alpha = \alpha_0 \cdots \alpha_{i-1} \lambda_1 \lambda_2 \alpha_{i+1} \cdots \alpha_{m-1}$ . Then  $\beta$  is fixed endpoint homotopic to  $\lambda\alpha$ ,  $[\lambda] \in \pi_1(\mathcal{U}, x)$ , and  $\alpha$  is associated with the new  $\mathcal{W}$ -chain.

Now suppose the next step of the simplicial homotopy is obtained by vertex deletion, say  $\dots, y_i, y_{i+1}, y_{i+2}, \dots$  to  $\dots, y_i, y_{i+2}, \dots$ . Then  $y_i, y_{i+2} \in W$  for some  $W \in \mathcal{W}$ . Join  $y_i$  to  $y_{i+2}$  by a path  $\lambda$  in  $W$ . Let  $\lambda = \alpha_0 \cdots \alpha_{i+1} \lambda^{-1} \alpha_{i-1}^{-1} \cdots \alpha_0^{-1}$  and  $\alpha = \alpha_0 \cdots \alpha_{i-1} \lambda \alpha_{i+2} \cdots \alpha_{m-1}$ . Then  $\beta$  is fixed endpoint homotopic to  $\lambda\alpha$ ,  $[\lambda] \in \pi_1(\mathcal{U}, x)$ , and  $\alpha$  is associated with the new  $\mathcal{W}$ -chain.

At the end of the simplicial homotopy the  $\mathcal{W}$ -chain is the trivial chain at  $x$ . Thus we have  $\beta$  is fixed endpoint homotopic to  $\lambda\alpha$  where  $[\lambda] \in \pi_1(\mathcal{U}, x)$  and  $\alpha$  is associated with the trivial chain. Thus  $\alpha$  is a loop based at  $x$  in some element of  $\mathcal{W}$  so  $[\lambda\alpha] \in \pi_1(\mathcal{U}, x)$ .  $\square$

*Remark 3.8.* The requirement of local path connectivity cannot be removed. In [8, Remark 4.9] the authors give an example of a path connected space that is lasso homotopically Hausdorff but not shape injective. The space is related to the space  $B$  in [5] (see Example 3.9). In fact the authors in [8] cite [12] which became [5].

**Example 3.9.** Let  $B'$  be the topologist's sine curve  $\{(x, \sin(1/(1-x))) : x \in (0, 1]\} \cup \{1\} \times [-1, 1]$  rotated about a vertical axis at the point  $(0, \sin(1))$ . This space is lasso homotopically Hausdorff (it is locally simply connected). Connecting arcs are used in [5] to obtain the Peano continuum  $B$ . Since  $B$  is not shape injective it cannot be lasso homotopically Hausdorff. A loop that goes around the outer annulus can be pulled in to the surface portion creating lassos.

## REFERENCES

- [1] W.A.Bogley, A.J.Sieradski, *Universal path spaces*. Preprint, 1998.
- [2] N. Brodskiy, J. Dydak, B. LaBuz, A. Mitra. *Rips Complexes and Covers in the Uniform Category*. Preprint, 2008.
- [3] N. Brodskiy, J. Dydak, B. LaBuz, A. Mitra. *Covering maps for locally path connected spaces*. Fund. Math. 218 (2012), 13-46.
- [4] J.W. Cannon, G.R. Conner, *On the fundamental groups of one-dimensional spaces*, Topology and its Applications 153 (2006), 2648–2672.
- [5] G. Conner, M. Meilstrup, D. Repovš, A. Zastrow, M. Željko. *On small homotopies of loops*. Topology Appl. 155 (2008), 1089–1097.
- [6] J. Dydak. *Partitions of Unity*. Topology Proceedings 27 (2003), 125–171.
- [7] K. Eda, K. Kawamura, *The fundamental group of one-dimensional spaces*. Topology and Its Applications 87 (1998), 163-172.
- [8] H. Fischer, A. Zastrow, *Generalized universal coverings and the shape group*. Fundamenta Mathematicae 197 (2007), 167–196.
- [9] H. Fischer, A. Zastrow, *The fundamental groups of subsets of closed surfaces inject into their first shape groups*. Algebraic and Geometric Topology 5 (2005) 1655-1676.
- [10] S. Mardesić, J. Segal, *Shape theory: the inverse limit approach*. North-Holland Mathematical Library 26, North-Holland, Amsterdam, 1982.
- [11] J. R. Munkres, *Topology*. Prentice Hall, Upper Saddle River, NJ 2000.

- [12] D. Repovš and A. Zastrow, *Shape injectivity is not implied by being strongly homotopically Hausdorff*. University of Ljubljana Institute of Mathematics, Physics and Mechanics Preprint series 43 (2005) No. 963.
- [13] E. Spanier, *Algebraic topology*, McGraw-Hill, New York 1966.
- [14] A. Zastrow, *Generalized  $\pi_1$ -determined covering spaces*. Unpublished, 2002 (revised version of 1996 notes).

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